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The Proper Vibration of Space: II

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Abstract

The proper vibrations of homogeneous and isotropic space are characterized by the autonomous conservative oscillation of spatial volume element. The conserved energy of oscillation is different from zero only when the space has positive curvature, and hence, it is closed. However, the oscillations of volume element of closed space imply the change in the signature of first metric form of space-time. This means that the metric actually has singular points.

1. *Introduction*

Finite space is in itself a natural and wall-less box which engenders atomicity of matter and light by the necessary discreteness of its proper modes of vibration (Edington, 1936; Schrödinger, 1937). For the finite space the different equations of mathematical physics which are of hyperbolic type provide eigenvalue problems (see, for example, Kulhánek, 1971, and references therein). The proper vibration of homogeneous and isotropic space itself, described in the terms of components of metric field of space-time, are considered by Kuhlánek (1971). There are the differential equations for oscillation of distance between two points and the spatial volume element derived from field equations.

The consequence of the field equation is the differential conservation law in the form of the general covariant continuity equation. It is well known that whenever the differential conservation law has the form mentioned above there is a conserved integral. Evaluation of this integral on a $t = \text{constant}$ hypersurface gives the result that when the space is finite the conserved energy of spatial volume element oscillation is different from zero. For space with zero and negative curvature this constant is zero.

Oscillation of the spatial volume element of closed space is mathematically possible. However, it necessarily leads to change in the signature of the line element of space-time. The signature is changed periodically from $+2$ to -2 during the evolution of the volume element, and there is a point where the scalar curvature of space-time becomes infinite. This means that the metric has singular points. It is necessary, however, to keep in mind that only by a fuller investigation of the field equations in the general case of a nonisotropic space would it be possible to answer the question whether this property of the solution is just special property of the isotropic closed space, not holding for the real case, where the isotropy of the space can only be approximate.

2. *Conserved Integral*

From the field equation (Kulhánek, 1966)

$$R_{ab} - \frac{R}{2} g_{ab} - \frac{\mathscr{H}^2}{h^2} g_{ab} = -\left(R + 4\frac{\mathscr{H}^2}{h^2}\right)(\tfrac{1}{2} g_{ab} - \dot{x}_a \dot{x}_b) \tag{2.1}$$

where $\mathscr{H}$ is the rest mass, h is Planck's constant and $\dot{x}^a$ are the components of the unit normal 4-vector to the 3-surface of the de Broglie's wave we have conservation equations

$$g^{bl}\left[\left(R + 4\frac{\mathscr{H}^2}{h^2}\right)(\tfrac{1}{2} g_{ab} - \dot{x}_a \dot{x}_b)\right]_{\|l} = 0 \tag{2.2}$$

Because of $\dot{x}^a \dot{x}^b g_{ab} = 1$ we have from (2.2) that

$$\frac{\partial}{\partial x^a}\left(\sqrt{(-g)}\sqrt{\left(R + 4\frac{\mathscr{H}^2}{h^2}\right)}\dot{x}^a\right] = 0 \tag{2.3}$$

Because of identity

$$\left[\sqrt{\left(R + 4\frac{\mathscr{H}^2}{h^2}\right)}\dot{x}^a\right]_{|a} \equiv \frac{1}{\sqrt{(-g)}}\frac{\partial}{\partial x^a}\left[\sqrt{(-g)}\sqrt{\left(R + 4\frac{\mathscr{H}^2}{h^2}\right)}\dot{x}^a\right] \tag{2.4}$$

we have from Kulhánek (1971) that

$$\left[\sqrt{\left(R + 4\frac{\mathscr{H}^2}{h^2}\right)}\dot{x}^a\right]_{|a} = 0 \tag{2.5}$$

The scalar $\sqrt{[R + 4(\mathscr{H}^2/h^2)]}$, called the index function (Synge, 1937), describes a property of space-time which remains conserved during the evolution of space-time. It is clear that the index function $\sqrt{[R + 4(\mathscr{H}^2/h^2)]}$ depends on the components of metric field and its first and second derivatives. If we integrate the identity (Kulhánek, 1966) over a 3-space on the boundaries of which $\sqrt{[R + 4(\mathscr{H}^2/h^2)]}\dot{x}^a$ vanishes, we get

$$\oint\left[\sqrt{\left(R + 4\frac{\mathscr{H}^2}{h^2}\right)}\dot{x}^a\right]_{|a}\sqrt{(-g)}\,dx^1\,dx^2\,dx^3 = \frac{d}{dx^4}\oint\sqrt{\left(R + 4\frac{\mathscr{H}^2}{h^2}\right)}\dot{x}^4\sqrt{(-g)}\,dx^1\,dx^2\,dx^3. \tag{2.6}$$

Now, because of (2.5) we have from (2.6) that the quantity

$$\oint \sqrt{\left(R + 4\frac{\mathscr{H}^2}{h^2}\right)} \dot{x}^4 \sqrt{(-g)}\, dx^1\, dx^2\, dx^3 \tag{2.7}$$

is a constant independent of x^4.

3. *Homogeneous and Isotropic Space*

If the unit normal 4-vector $\dot{x}^a$ has components (0,0,0,1) and we use the Robertson–Walker metric

$$ds^2 = (dx^0)^2 - U^2(x^0)\, dw^2 \tag{3.1}$$

where $U(x^0)$ is an arbitrary function and

$$dw^2 = \frac{1}{1 + k(r^2/4)}(dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta\, d\phi^2)) \tag{3.2}$$

defines the 3-space of constant curvature k in which $dw^2 > 0$ for any two points, then, from (2.7) we obtain

$$\sqrt{\left(R + 4\frac{\mathscr{H}^2}{h^2}\right)} U^3 \int \frac{r^2 \sin^2\vartheta\, d\vartheta\, d\phi\, dr}{[1 + k(r^2/4)]^3} = \text{constant.} \tag{3.3}$$

Here, $\sqrt{[R + 4(\mathscr{H}^2/h^2)]}$ and U^3 are functions of coordinate x^0 only and $\mathscr{H}^2 = \mathscr{H}^2$ is the rest mass.

Denoting the quantity

$$\int \frac{r^2 \sin^2\vartheta\, d\vartheta\, d\phi\, dr}{[1 + k(r^2/4)]^3} \tag{3.4}$$

by I we can transcribe (3.3) in the form

$$\sqrt{\left(R + 4\frac{\mathscr{H}^2}{h^2}\right)} U^3 = \frac{\text{constant}}{I} \tag{3.5}$$

On the other hand, from (2.3) we have

$$\frac{d}{dx^0}\left[U^3 \sqrt{\left(R + 4\frac{\mathscr{H}^2}{h^2}\right)}\right] = 0$$

and hence,

$$U^6\left(R + 4\frac{\mathscr{H}^2}{h^2}\right) = \beta^2 \tag{3.6}$$

where β^2 is an integrating constant, which has been written by Kulhánek (1971) as $\beta^2 = -\frac{4}{3}W$, where W is the conserved energy of volume oscillation. On comparing (3.6) with (3.5) we see that $\beta^2 = \text{constant}/I$. Hence, the constants W and β^2 depend on the quantity I (3.5) which, as simple computation shows, is infinite for $k=0$ and $k<0$, and is finite for $k>0$. Thus, writing

$$W = -\frac{3}{4}\frac{\text{constant}}{I} \tag{3.7}$$

we see that the energy of volume oscillation is zero for Euclidean and pseudo-spherical space. In both these cases the space-time has constant curvature $R = -4(\mathscr{H}^2/h^2)$.

4. *Oscillations of Spatial Volume Element*

From the form (3.1) follows that spatial volume element is proportional to U^3. If we denote $V \equiv U^3$ then from field equation (2.1) we can write (Kulhánek, 1971) that

$$\tfrac{1}{2}\dot{V}^2 + \tfrac{3}{2}\frac{\mathscr{H}^2}{h^2}V^2 + \tfrac{9}{2}kV^{4/3} \equiv W \tag{4.1}$$

The results of the foregoing section are leading us to consider the integration of (4.1) for three different cases: $k = W = 0$; $k<0$, $W=0$; $k>0$, $W \neq 0$. The case when $k=0$ and $W=0$ was, as a mathematically simple case, considered by Kulhánek (1971). The second case, $k<0$, $W=0$, is considered by Kulhánek (1971) under the case $k \neq 0$, $W=0$. It is shown there that for real solution V of equation (4.1) we necessarily must have $k<0$. So what remains is the case $k>0$, $W \neq 0$. The relation $V \equiv U^3$ enables us to rewrite (4.1) as

$$U^4\left[\left(\frac{dU}{dx^0}\right)^2 + \frac{1}{3}\frac{\mathscr{H}^2}{h^2}U^2 + K\right] = \tfrac{2}{9}W \tag{4.2}$$

Introducing a new independent variable $d\tau = U^{-1}dx^0$ and writing $U^2 \equiv y$ from (4.2) we obtain

$$\frac{1}{2}\left(\frac{dy}{d\tau}\right)^2 + \frac{2}{l^2}y^2 + \frac{2}{3}\frac{\mathscr{H}^2}{h^2}y^3 = \tfrac{4}{9}W \tag{4.3}$$

where we put $k = 1/l^2 > 0$. After differentiation, from (4.3), we get

$$\frac{d^2y}{d\tau^2} + \frac{4}{l^2}y + 2\frac{\mathscr{H}^2}{h^2}y^2 = 0 \tag{4.4}$$

The solution of (4.4) is discussed, for example, by Bradbury (1968). We can write

$$y = B + A\,sn^2(a\tau) \tag{4.5}$$

where A, B and a are parameters and sn is a Jacobi function of modulus k. The potential function, as we see immediately from (4.3) is

$$\psi(y) = \frac{2}{l^2}y^2 + \frac{2}{3}\frac{\mathscr{H}^2}{h^2}y^3 \tag{4.6}$$

Since the potential function is not symmetric, the average spatial volume element is not zero. The constant B in (4.5) provides a means of adjusting the average spatial volume element properly depending on the energy W. The location and value of the relative maximum of $\psi(y)$ are

$$y_1 = -2\frac{h^2}{l^2\mathscr{H}^2}; \qquad \psi_1 = \frac{8}{3}\frac{h^4}{l^6\mathscr{H}^4} \tag{4.7}$$

The potential ψ has the value ψ_1 also at $y = h^2 l^{-2}\mathscr{H}^{-2}$, and the zeros of ψ are at $y = 0$ and $y = \frac{3}{2}y_1$. The turning points of periodic motion occur, as we see from (4.5), when $sn^2(a\tau) = 1$ and $sn^2(a\tau) = 0$, and are

$$y_2 = A + B \qquad \text{and} \qquad y_3 = B$$

As the energy W approaches the limiting value ψ_1, the turning points approach

$$y_2 = y_1 \quad \text{or} \quad -\tfrac{1}{2}y_1 \qquad \text{and} \qquad y_3 = -\tfrac{1}{2}y_1 \quad \text{or} \quad y_1 \tag{4.8}$$

The limiting values of parameters A and B, therefore, must be

$$B = \frac{h^2}{l^2\mathscr{H}^2} \quad \text{or} \quad -2\frac{h^2}{l^2\mathscr{H}^2}; \qquad A = -3\frac{h^2}{l^2\mathscr{H}^2} \quad \text{or} \quad 3\frac{h^2}{l^2\mathscr{H}^2} \tag{4.9}$$

Thus we see that function (4.5) is changing its sign. With the same notation as above we can rewrite the element (3.1) as

$$ds^2 = y(d\tau^2 + dw^2)$$

and we see that y is the conforming factor between the space-time with line element (3.1) and space-time with line element $d\bar{s}^2 = d\tau^2 - dw^2$. The change of sign of y means the change of signature of line elements (3.1). This means that the periodical changes of spatial volume element of closed space are accompanied with periodical change of signature of space-time line element.

5. *Conclusion*

The results of this paper, together with those of an earlier publication (Kulhánek, 1971), show that, in the case of a homogeneous and isotropical space periodic changes of spatial volume element are possible, without any difficulties, only when the space has negative curvature. However, this space is not closed and we cannot expect any result which will demonstrate discreteness of its vibration.

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